UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

**FURTHER MATHEMATICS** **9231/13**

Paper 1 **October/November 2012**

**3 hours**

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

*1588720257*

**READ THESE INSTRUCTIONS FIRST**

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.



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**[Turn over**

**1** Show that $\sum_{r=n+1}^{2n} r^2 = \frac{1}{6}n(2n+1)(7n+1)$. [4]

**2** Find the set of values of $a$ for which the system of equations

$$\begin{aligned} ax + y + 2z &= 0, \\ 3x - 2y &= 4, \\ 3x - 4y - 6az &= 14, \end{aligned}$$

has a unique solution. [4]

**3** Let $S_N = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{N}{(N+1)!}$. Prove by mathematical induction that, for all positive integers $N$,

$$S_N = 1 - \frac{1}{(N+1)!}.$$ [5]

**4** The points $A$, $B$ and $C$ have position vectors $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$ and $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ respectively. Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]

Deduce, in either order, the exact value of

**(i)** the area of the triangle $ABC$,

**(ii)** the perpendicular distance from $C$ to $AB$.

[3]

**5** The curve $C$ has polar equation $r = 1 + 2\cos\theta$. Sketch the curve for $-\frac{2}{3}\pi \leqslant \theta < \frac{2}{3}\pi$. [2]

Find the area bounded by $C$ and the half-lines $\theta = -\frac{1}{3}\pi$, $\theta = \frac{1}{3}\pi$. [4]

**6** The curve $C$ has parametric equations

$$x = t^2, \quad y = \frac{1}{4}t^4 - \ln t,$$

for $1 \leqslant t \leqslant 2$. Find the area of the surface generated when $C$ is rotated through $2\pi$ radians about the $y$-axis. [7]

**7** A cubic equation has roots $\alpha$, $\beta$ and $\gamma$ such that

$$\begin{aligned} \alpha + \beta + \gamma &= 4, \\ \alpha^2 + \beta^2 + \gamma^2 &= 14, \\ \alpha^3 + \beta^3 + \gamma^3 &= 34. \end{aligned}$$

Find the value of $\alpha\beta + \beta\gamma + \gamma\alpha$. [2]

Show that the cubic equation is

$$x^3 - 4x^2 + x + 6 = 0,$$

and solve this equation. [6]

**8** Let $z = \cos\theta + \mathrm{i}\sin\theta$. Show that

$$1 + z = 2\cos\tfrac{1}{2}\theta(\cos\tfrac{1}{2}\theta + \mathrm{i}\sin\tfrac{1}{2}\theta).$$ [3]

By considering $(1+z)^n$, where $n$ is a positive integer, deduce the sum of the series

$$\binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \ldots + \binom{n}{n}\sin n\theta.$$ [6]

**9** The curve $C$ has equation $y = \dfrac{x^2 - 3x + 3}{x - 2}$. Find the equations of the asymptotes of $C$. [3]

Show that there are no points on $C$ for which $-1 < y < 3$. [4]

Find the coordinates of the turning points of $C$. [3]

Sketch $C$. [2]

**10** The curve $C$ has equation $x^3 + y^3 = 3xy$, for $x > 0$ and $y > 0$. Find a relationship between $x$ and $y$ when $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$. [4]

Find the exact coordinates of the turning point of $C$, and determine the nature of this turning point. [8]

**11** Show that $\displaystyle\int x(1 - x^2)^{\frac{1}{2}}\,\mathrm{d}x = -\tfrac{1}{3}(1 - x^2)^{\frac{3}{2}} + c$, where $c$ is a constant. [1]

Given that $I_n = \displaystyle\int_0^1 x^n(1 - x^2)^{\frac{1}{2}}\,\mathrm{d}x$, prove that, for $n \geqslant 2$,

$$(n + 2)I_n = (n - 1)I_{n-2}.$$ [5]

Use the substitution $x = \sin u$ to show that

$$\int_0^1 (1 - x^2)^{\frac{1}{2}}\,\mathrm{d}x = \tfrac{1}{4}\pi.$$ [5]

Find $I_4$. [2]

**12** Answer only **one** of the following two alternatives.

**EITHER**

The vector $\mathbf{e}$ is an eigenvector of each of the $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$, with corresponding eigenvalues $\lambda$ and $\mu$ respectively. Prove that $\mathbf{e}$ is an eigenvector of the matrix $\mathbf{AB}$ with eigenvalue $\lambda\mu$. [2]

It is given that the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -2 & -2 \\ 1 & 2 & 2 \end{pmatrix},$$

has eigenvectors $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Find the corresponding eigenvalues. [2]

Given that 2 is also an eigenvalue of $\mathbf{A}$, find a corresponding eigenvector. [2]

The matrix $\mathbf{B}$, where

$$\mathbf{B} = \begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & 2 \\ -3 & -6 & -6 \end{pmatrix},$$

has the same eigenvectors as $\mathbf{A}$. Given that $\mathbf{AB} = \mathbf{C}$, find a non-singular matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{P}^{-1}\mathbf{C}^2\mathbf{P} = \mathbf{D}.$$ [8]

**OR**

Obtain the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 13x = 75\cos 2t.$$ [7]

Given that $x = 5$ and $\frac{dx}{dt} = 0$ when $t = 0$, find $x$ in terms of $t$. [4]

Show that, for large positive values of $t$ and for any initial conditions,

$$x \approx 5\cos(2t - \phi),$$

where the constant $\phi$ is such that $\tan\phi = \frac{4}{3}$. [3]